TRINITY COLLEGE

Semester Two Examination, 2020

Question/Answer booklet

MATHEMATICS SPECIALIST UNITS 1&2

Section One: Calculator-free

SOLUTIONS

WA student number: In figures

In words

Your name

Time allowed for this section

Reading time before commencing work: five minutes
Working time: fifty minutes

Number of additional answer booklets used (if applicable):

Materials required/recommended for this section

To be provided by the supervisor
This Question/Answer booklet
Formula sheet

To be provided by the candidate
Standard items: pens (blue/black preferred), pencils (including coloured), sharpener, correction fluid/tape, eraser, ruler, highlighters

Special items: nil

Important note to candidates

No other items may be taken into the examination room. It is **your** responsibility to ensure that you do not have any unauthorised material. If you have any unauthorised material with you, hand it to the supervisor **before** reading any further.

Structure of this paper

Section	Number of questions available	Number of questions to be answered	Working time (minutes)	Marks available	Percentage of examination
Section One: Calculator-free	8	8	50	52	35
Section Two: Calculator-assumed	13	13	100	98	65
				Total	100

Instructions to candidates

1. The rules for the conduct of Trinity College examinations are detailed in the *Instructions to Candidates* distributed to students prior to the examinations. Sitting this examination implies that you agree to abide by these rules.

2. Write your answers in this Question/Answer booklet preferably using a blue/black pen. Do not use erasable or gel pens.

3. You must be careful to confine your answers to the specific question asked and to follow any instructions that are specific to a particular question.

4. Show all your working clearly. Your working should be in sufficient detail to allow your answers to be checked readily and for marks to be awarded for reasoning. Incorrect answers given without supporting reasoning cannot be allocated any marks. For any question or part question worth more than two marks, valid working or justification is required to receive full marks. If you repeat any question, ensure that you cancel the answer you do not wish to have marked.

5. It is recommended that you do not use pencil, except in diagrams.

6. Supplementary pages for planning/continuing your answers to questions are provided at the end of this Question/Answer booklet. If you use these pages to continue an answer, indicate at the original answer where the answer is continued, i.e. give the page number.

7. The Formula sheet is not to be handed in with your Question/Answer booklet.

Section One: Calculator-free **35% (52 Marks)**

This section has **eight** questions. Answer **all** questions. Write your answers in the spaces provided.

Working time: 50 minutes.

Question 1 **(7 marks)**

Two matrices are $A = \begin{bmatrix} -2 & 2 \\ -4 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & -5 \\ 10 & 5 \end{bmatrix}$. Determine

(a) $3A - 2B$. (2 marks)

Solution

$$3A - 2B = \begin{bmatrix} -6 & 6 \\ -12 & 9 \end{bmatrix} - \begin{bmatrix} 0 & -10 \\ 20 & 10 \end{bmatrix} = \begin{bmatrix} -6 & 16 \\ -32 & -1 \end{bmatrix}$$

Specific behaviours

✓ one correct multiple
✓ correct matrix

(b) A^{-1}. (2 marks)

Solution

$$|A| = (-2)(3) - (-4)(2) = 2$$

$$A^{-1} = \frac{1}{2}\begin{bmatrix} 3 & -2 \\ 4 & -2 \end{bmatrix} = \begin{bmatrix} 1.5 & -1 \\ 2 & -1 \end{bmatrix}$$

Specific behaviours

✓ determinant
✓ correct matrix

(c) $AB + B^2$. (3 marks)

Solution

$$(A + B) \times B = \begin{bmatrix} -2 & -3 \\ 6 & 8 \end{bmatrix} \times \begin{bmatrix} 0 & -5 \\ 10 & 5 \end{bmatrix} = \begin{bmatrix} -30 & -5 \\ 80 & 10 \end{bmatrix}$$

Specific behaviours

✓ factors
✓ $A + B$
✓ correct matrix

Alternative Solution

$$AB = \begin{bmatrix} 20 & 20 \\ 30 & 35 \end{bmatrix}$$

$$B^2 = \begin{bmatrix} -50 & -25 \\ 50 & -25 \end{bmatrix}$$

$$AB + B^2 = \begin{bmatrix} -30 & -5 \\ 80 & 10 \end{bmatrix}$$

Specific behaviours

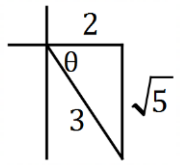
✓ AB
✓ B^2
✓ correct matrix

Question 2 **(6 marks)**

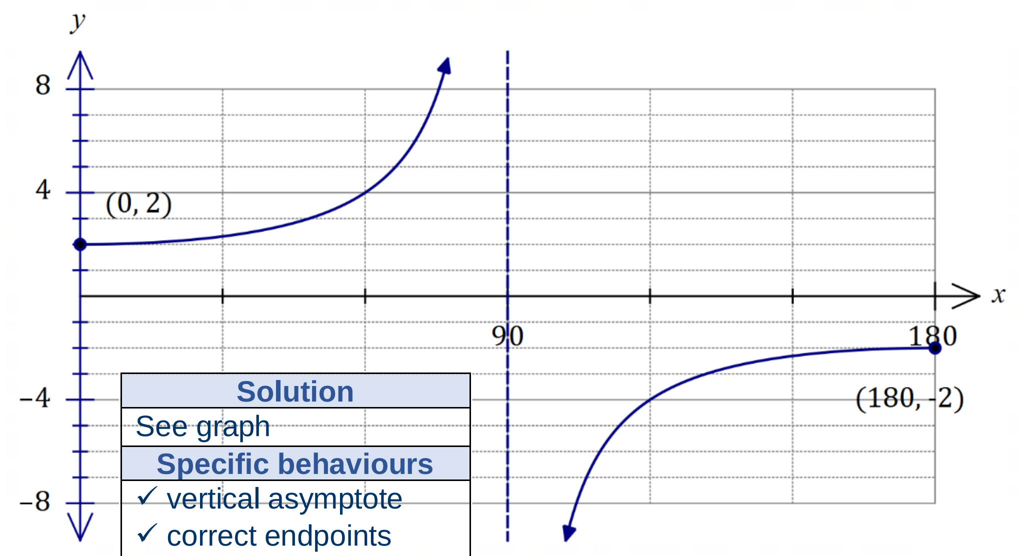
(a) State the exact value of $\cot 60°$. (1 mark)

Solution
$\dfrac{1}{\tan 60°} = \dfrac{1}{\sqrt{3}} = \dfrac{\sqrt{3}}{3}$
Specific behaviours
✓ correct value

(b) Given that $\sec\theta = \dfrac{3}{2}$ and $-90° \le \theta \le 0°$, state the exact value of $\operatorname{cosec}\theta$. (2 marks)

Solution
 $\operatorname{cosec}\theta = \dfrac{3}{-\sqrt{5}} = -\dfrac{3\sqrt{5}}{5}$
Specific behaviours
✓ value is negative
✓ correct value

(c) Sketch the graph of $y = 2\sec x$ for $0° \le x \le 180°$ on the axes below. (3 marks)



Solution
See graph
Specific behaviours
✓ vertical asymptote
✓ correct endpoints
✓ smooth curves

Question 3 **(8 marks)**

(a) Express $\left(\sqrt{5}+\sqrt{-5}\right)^2$ in the form $a+bi$ where $a, b \in \mathbb{R}$. (2 marks)

Solution
$\left(\sqrt{5}+\sqrt{-5}\right)\left(\sqrt{5}+\sqrt{-5}\right) = 5 + 2\sqrt{5}\sqrt{5}i - 5$ $= 10i$
Specific behaviours
✓ uses $i^2 = -1$ ✓ correctly simplifies

(b) Two complex numbers are $u = 8 + i$ and $v = 2 - i$. Calculate

(i) $u \times v$. (1 mark)

Solution
$uv = (8+i)(2-i)$ $= 17 - 6i$
Specific behaviours
✓ correct product

(ii) $u \div v$. (2 marks)

Solution
$\frac{u}{v} = \frac{8+i}{2-i} \times \frac{2+i}{2+i}$ $= \frac{15+10i}{5}$ $= 3 + 2i$
Specific behaviours
✓ correctly uses conjugate of v ✓ correct quotient, simplified

(iii) $\text{Im}(2\bar{v} - iu)$. (3 marks)

Solution
$2\bar{v} = 2(2+i) = 4 + 2i$ $iu = i(8+i) = -1 + 8i$ $2\bar{v} - iu = 5 - 6i \Rightarrow \text{Im}(2\bar{v} - iu) = -6$
Specific behaviours
✓ $2\bar{v}$ ✓ product iu ✓ imaginary part of difference

Question 4 **(7 marks)**

(a) Prove that $\tan 3A = \dfrac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$. (4 marks)

Solution
$$\tan(2A + A) = \frac{\tan 2A + \tan A}{1 - \tan 2A \tan A}$$ $$\tan 3A = \frac{\dfrac{2\tan A}{1-\tan^2 A} + \tan A}{1 - \dfrac{2\tan A}{1-\tan^2 A} \times \tan A}$$ $$= \frac{2\tan A + \tan A - \tan^3 A}{1 - \tan^2 A - 2\tan^2 A}$$ $$= \frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A}$$
Specific behaviours
✓ uses angle sum identity ✓ uses double angle identity ✓ simplifies ✓ simplifies

(b) Solve $3\tan A - \tan^3 A = 1 - 3\tan^2 A$ for $0° < A < 180°$. (3 marks)

Solution
$$\frac{3\tan A - \tan^3 A}{1 - 3\tan^2 A} = 1$$ $$\tan 3A = 1$$ $$A = 15°, 75°, 135°$$
Specific behaviours
✓ writes equation using triple angle ✓ correct 1st solution ✓ all correct solutions

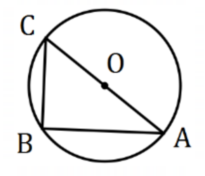
Question 5 **(5 marks)**

AC is a diameter of a circle centre O and point B lies on the circumference of the circle.

Let $\overrightarrow{OA} = \mathbf{a}$ and $\overrightarrow{OB} = \mathbf{b}$.

Use a vector method to prove that $\angle ABC = 90°$.

Solution



$$\overrightarrow{AB} = \mathbf{b} - \mathbf{a}, \qquad \overrightarrow{CB} = \mathbf{a} + \mathbf{b}$$

$$\begin{aligned}\overrightarrow{CB} \cdot \overrightarrow{AB} &= (\mathbf{a} + \mathbf{b}) \cdot (\mathbf{b} - \mathbf{a}) \\ &= \mathbf{a} \cdot \mathbf{b} - \mathbf{a} \cdot \mathbf{a} + \mathbf{b} \cdot \mathbf{b} - \mathbf{b} \cdot \mathbf{a} \\ &= |\mathbf{b}|^2 - |\mathbf{a}|^2 \\ &= 0 \quad (|\mathbf{a}| = |\mathbf{b}| = \text{radius})\end{aligned}$$

Hence $\overrightarrow{AB}$ and $\overrightarrow{CB}$ are perpendicular ($\angle ABC = 90°$) since their magnitudes are not zero yet the scalar product is zero.

Specific behaviours

✓ labelled sketch
✓ vectors $\overrightarrow{AB}$ and $\overrightarrow{CB}$
✓ forms scalar product
✓ simplifies
✓ explains result

Question 6 **(6 marks)**

(a) Determine the equation of the real quadratic $f(z)$ in the form $z^2 + az + b$ given that $f(5 - 3i) = 0$. (2 marks)

Solution
$f(z) = (z - 5 + 3i)(z - 5 - 3i)$ $= z^2 - 10z + 34$
Specific behaviours
✓ shows product of linear factors
✓ correct equation

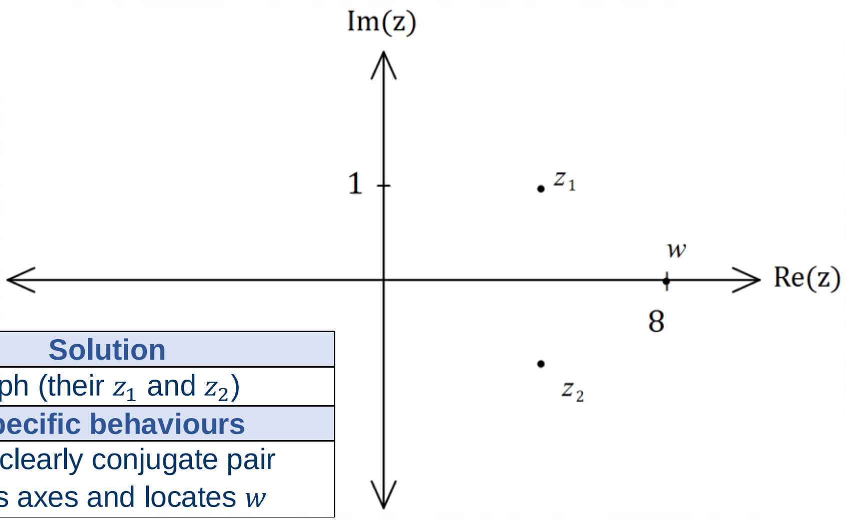
Alternative Solution
$a = -2(5) = -10$ $b = 5^2 + 3^2 = 34$ $f(z) = z^2 - 10z + 34$
Specific behaviours
✓ shows sum and product of roots
✓ correct equation

(b) Let $g(z) = z^2 - 8z + 17$.

(i) Determine z_1 and z_2, the complex roots of g. (2 marks)

Solution
$(z - 4)^2 = -17 + 4^2$ $= -1 = i^2$ $z = 4 \pm i$ $z_1 = 4 + i, \quad z_2 = 4 - i$
Specific behaviours
✓ indicates suitable method
✓ correct conjugate roots

(ii) Sketch and label z_1, z_2 and $w = z_1 + z_2$ in the complex plane below. (2 marks)



Solution
See graph (their z_1 and z_2)
Specific behaviours
✓ roots clearly conjugate pair
✓ scales axes and locates w

Question 7 **(7 marks)**

Use mathematical induction to prove the following proposition $P(n)$ for every integer $n \geq 0$.

$$P(n):\ 1 + 9 + 17 + 25 + \cdots + (8n + 1) = (n + 1)(4n + 1)$$

Solution
1. When $n = 0$: $P(0): 8(0) + 1 = (0 + 1)(4(0) + 1) \rightarrow 1 = 1, \therefore \text{true}$ 2. Assume $P(k)$ is true: $1 + 9 + \cdots + (8k + 1) = (k + 1)(4k + 1)$ 3. Now required to prove $P(k + 1)$: $1 + 9 + \cdots + (8k + 1) + (8(k + 1) + 1) = \big((k + 1) + 1\big)(4(k + 1) + 1)$ $= (k + 2)(4k + 5)$ From assumption: $1 + 9 + \cdots + (8k + 1) = (k + 1)(4k + 1)$ Hence $1 + 9 + \cdots + (8k + 1) + (8(k + 1) + 1) = (k + 1)(4k + 1) + (8(k + 1) + 1)$ $= 4k^2 + 5k + 1 + 8k + 9$ $= 4k^2 + 13k + 10$ $= (k + 2)(4k + 5)$ Since $P(0)$ is true and we have shown that $P(k + 1)$ is true if $P(k)$ is true then $P(n)$ is true for all integers $n \geq 0$.

Specific behaviours
✓ shows true for $n = 0$ ✓ states assumption that $P(k)$ is true ✓ indicates expression required for $P(k + 1)$ ✓ adds $(k + 1)^{th}$ term to assumption ✓ expands RHS ✓ factors RHS ✓ uses principle of mathematical induction

Question 8 **(6 marks)**

(a) Determine the vector projection of $2\mathbf{i} - \mathbf{j}$ on $-3\mathbf{i} + 4\mathbf{j}$. (2 marks)

Solution
$\left(\frac{1}{5}\begin{pmatrix}-3\\4\end{pmatrix}\cdot\begin{pmatrix}2\\-1\end{pmatrix}\right)\frac{1}{5}\begin{pmatrix}-3\\4\end{pmatrix} = \frac{-10}{25}\begin{pmatrix}-3\\4\end{pmatrix}$ $= \begin{pmatrix}6/5\\-8/5\end{pmatrix}$
Specific behaviours
✓ indicates method ✓ correct projection

Alternative Solution
$\frac{\begin{pmatrix}2\\-1\end{pmatrix}\cdot\begin{pmatrix}-3\\4\end{pmatrix}}{\begin{pmatrix}-3\\4\end{pmatrix}\cdot\begin{pmatrix}-3\\4\end{pmatrix}}\begin{pmatrix}-3\\4\end{pmatrix} = \frac{-10}{25}\begin{pmatrix}-3\\4\end{pmatrix}$ $= \begin{pmatrix}6/5\\-8/5\end{pmatrix}$
Specific behaviours
✓ indicates method ✓ correct projection

(b) The vectors $a\mathbf{i} + \mathbf{j}$ and $4\mathbf{i} + b\mathbf{j}$ are perpendicular and the sum of their magnitudes is 10. Determine the values of the constants a and b. (4 marks)

Solution
Perpendicular: $4a + b = 0$. Magnitudes: $\sqrt{a^2+1} + \sqrt{16+b^2} = 10$. Hence $b = -4a \Rightarrow b^2 = 16a^2$ $\sqrt{a^2+1} + \sqrt{16+16a^2} = 10$ $\sqrt{a^2+1} + 4\sqrt{a^2+1} = 10$ $\sqrt{a^2+1} = 2$ $a^2 = 3 \Rightarrow a = \pm\sqrt{3}$ $b = \mp 4\sqrt{3}$ Hence $a = \sqrt{3}, b = -4\sqrt{3}$ or $a = -\sqrt{3}, b = 4\sqrt{3}$.
Specific behaviours
✓ two equations ✓ eliminates one variable ✓ solves for one variable ✓ clearly states both solutions

End of questions

Supplementary page

Question number: _________